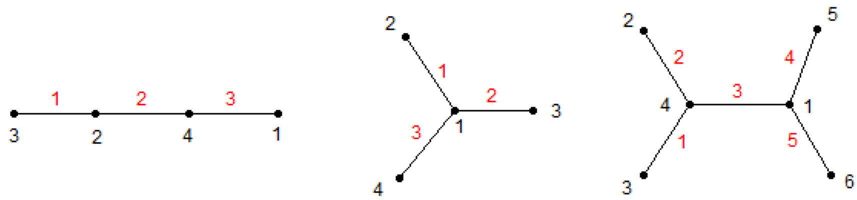
Graceful Trees

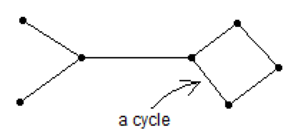
Leader Checklist

- ☐ Read through the module. (Yup, that's all!)

- Look at these diagrams. What do you observe?



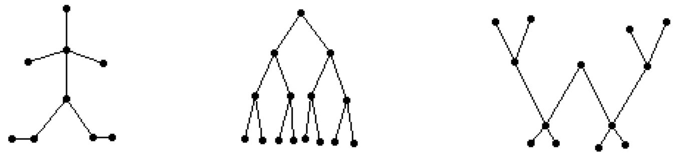
- These are examples of *trees*. They consist of a bunch of dots (*vertices*[1]) that are joined by line segments (*edges*). More generally, a *graph* is a collection of vertices, some of which are joined by edges. A tree is a special kind of graph that doesn't contain any *cycles* – paths you can travel on that return to their starting point. For contrast, the graph to the right is not a tree, because it contains a cycle.



- Notice that we're using the word "graph" differently from how you've used it in the past!
  - You probably think of a graph as being the curve you get when you plot input-output pairs of a function, with respect to an *x*-axis and a *y*-axis.
  - But there's another use of the word "graph," which comes from an area of mathematics called (wait for it!) Graph Theory.
  - In this context, graphs represent how collections of things are connected. For example, you could model your network of friends on social media using a graph, representing each person with a vertex and joining up two vertices when they are connected online. Or you could use a graph to model connections in a city's electrical grid or water supply.
  - In addition to their applications to real world problem solving, graphs are super cool, and mathematicians study them just for fun!

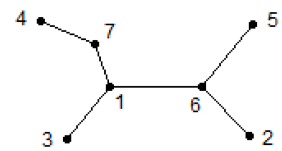
[1] The singular is *vertex*, and the plural is *vertices*.

- Here are some trees. In each one, count the number of edges, and count the number of vertices.

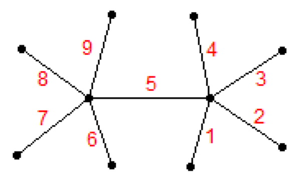


Notice that every tree has one more vertex than it has edges. A *graceful labeling* of a tree is a way to label all of its edges with the numbers 1 to *n*, and all of its vertices with the numbers 1 to $n+1$, so that every edge number is the difference of its two adjacent vertex numbers. (We're using *n* for the number of edges in the tree.) Be careful not to repeat – once you've used a number for an edge, you can't use it for another edge, and the same with vertices.

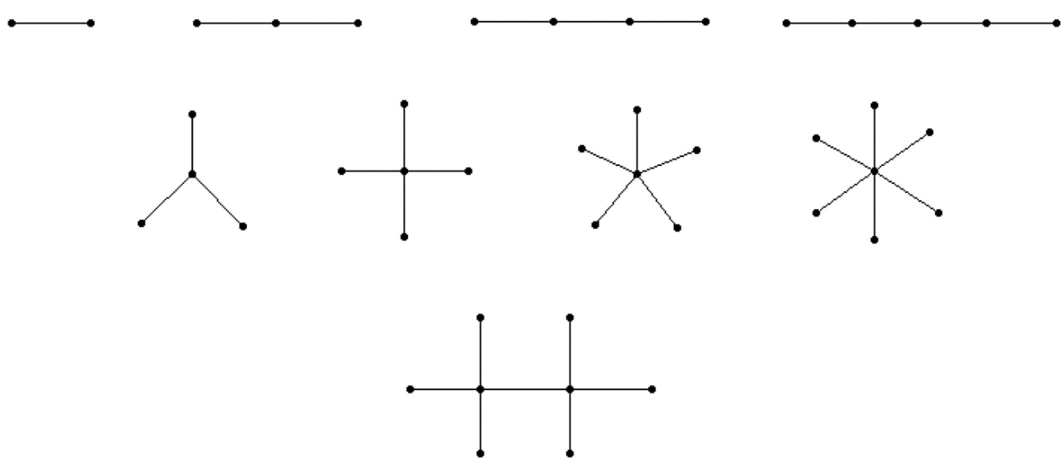
- Can you finish the graceful labeling of this tree? We've put in the vertex labels, so all you need are the edge labels.



- How about this one? This time, you need to find the vertex labels.



- Can you find graceful labelings of these trees?



- Follow-up:
  - Look online or in math books for more information on graph theory. The book *Pearls in Graph Theory*, by Nora Hartsfield and Gerhard Ringel, is a good one.
  - Do all trees have graceful labelings? We don't know! The *Graceful Trees Conjecture* says that they do, but it hasn't been proved or disproved yet. This conjecture is due to two mathematicians, Gerhard Ringel and Anton Kotzig. If you can prove this conjecture, or find a counterexample to it, you would be famous!